

SEPARABLE ELASTIC BANACH SPACES ARE UNIVERSAL

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To the memory of Edward Odell

ABSTRACT. A Banach space X is *elastic* if there is a constant K so that whenever a Banach space Y embeds into X , then there is an embedding of Y into X with constant K . We prove that $C[0, 1]$ embeds into separable infinite dimensional elastic Banach spaces, and therefore they are universal for all separable Banach spaces. This confirms a conjecture of Johnson and Odell. The proof uses incremental embeddings into X of $C(K)$ spaces for countable compact K of increasing complexity. To achieve this we develop a generalization of Bourgain's basis index that applies to unconditional sums of Banach spaces and prove a strengthening of the weak injectivity property of these $C(K)$ that is realized on special reproducible bases.

1. INTRODUCTION

A Banach space X is K -*elastic* provided that if a Banach space Y embeds into X then Y must K -embed into X . That is, there is an isomorphism T from Y into X with

$$\|y\| \leq \|Ty\| \leq K\|y\|$$

for all $y \in Y$. X is called *elastic* if it is K -elastic for some $K < \infty$. The space $C[0, 1]$ is 1-elastic simply because it is universal; every separable Banach space 1-embeds into $C[0, 1]$. Thus if a separable Banach space X contains an isomorphic copy of $C[0, 1]$, then X is elastic. Johnson and Odell conjectured that such spaces are the only separable elastic Banach spaces. In this paper we prove this conjecture.

Theorem 1. *Let X be a separable infinite dimensional elastic Banach space. Then $C[0, 1]$ is isomorphic to a subspace of X .*

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Johnson and Odell introduced the notion of an elastic space in [4] where it plays a pivotal role in their proof of the main result that for infinite dimensional separable Banach spaces *the diameter in the Banach-Mazur distance of the isomorphism class of a space is infinite*. This is derived as an immediate consequence of the following.

Theorem 2 (Johnson-Odell). *If X is a separable Banach space and there is a K so that every isomorph of X is K -elastic, then X is finite-dimensional.*

The reason for the conjecture is that, as they noted, Theorem 2 would be an immediate consequence of Theorem 1. We outline this argument below.

Proof of Theorem 2. In [6] it is shown there that there are equivalent norms that ‘arbitrarily distort’ the usual norm of $C[0, 1]$, that is, for all n , there exists an equivalent norm $|\cdot|_n$ on $C[0, 1]$ so that the best embedding constant of $C[0, 1]$ with the usual norm into $(C[0, 1], |\cdot|_n)$ is greater than n . Let X be as in the hypothesis. For each n , X is isomorphic to a subspace of $C[0, 1]$ with norm $|\cdot|_n$ and is K -elastic with this norm. If X contains a subspace Y which is isomorphic to $C[0, 1]$, $C[0, 1]$ with its usual norm is K -isomorphic to a subspace of $(X, |\cdot|_n)$, and consequently, of $(C[0, 1], |\cdot|_n)$. This is a contradiction for n large enough. \square

One of the main steps in [4] in proving Theorem 2 is to show that an elastic space must contain a nice space.

Theorem 3 (Johnson-Odell). *Let X be elastic, separable and infinite dimensional. Then c_0 is isomorphic to a subspace of X .*

The proof of this theorem uses Bourgain’s basis index and a clever transfinite induction argument. Our proof of Theorem 1 follows the same general outline as their argument. The main machinery (Proposition 15) in our proof is to show that whenever a sequence of $C(K_n)$ spaces embed into X where each K_n is countable and compact, then $\left(\sum_{n=1}^{\infty} C(K_n)\right)_{c_0}$ embeds into X . This can be seen as a higher dimensional analogue of Theorem 3. However, to be able to carry out such an extension by generalizing the proof given by Johnson and Odell one faces two rather fundamental problems. The first is that one needs to be able to do a Bourgain basis index argument for a basis of the c_0 -sum. There does not seem to be a feasible way of doing so in this setting since no basis has such a simple homogeneous structure as the usual basis of c_0 itself has. We solve this problem by not working with the basis

index but rather developing an ordinal index for unconditional sums of Banach spaces. This is done in Section 3 and may be of independent interest. The second major problem is that the proof of Theorem 3 requires embedding a countable family of incrementally renormed spaces Y_α into an elastic space as spans of blocks. Working with sums of infinite dimensional spaces requires replacing blocks by well positioned subspaces. To be able to ‘dig ourselves out of this hole’ by patching together Y_α spaces of the stage α , one needs copies of these spaces to be well complemented with nice projections that have rather large kernels. This is achieved by a strengthening of a remarkable theorem of Pelczynski that separable $C(K)$ spaces are *weak injective* [9] (See also [10, Theorem 3.1].), and of a useful observation due to Lindenstrauss and Pelczynski that $C(K)$ spaces have *reproducible bases* [6]. These are addressed in Section 2. The proofs of the main machinery (Proposition 15) and Theorem 1 are given in Section 4. Once Proposition 15 is proved, Theorem 1 is easily deduced using a well known theorem of Bourgain. (See [8, Proposition 2.3].)

Theorem 4 (Bourgain). *If X is universal for the class of spaces $C(K)$ where K is countable compact metric, then X contains an isomorphic copy of $C[0, 1]$.*

2. COMPLEMENTABLY REPRODUCIBLE BASES OF $C(\omega^\alpha)$

In this section we give strengthenings of two important properties of $C(K)$ spaces that are instrumental for the proof of the main result.

The first property, due to Lindenstrauss and Pelczynski [6, Theorem 4.3], asserts in particular that the canonical bases (explained below) of $C(K)$ spaces for countable compact K are *reproducible*. A basis (x_n) is reproducible with constant K if whenever $[(x_n)]$ is isometrically embedded into a space X with a basis, one can find, for every $\epsilon > 0$, a block basis in X that is $(K + \epsilon)$ -equivalent to (x_n) . In our variant, the connection between the isomorphism and the block basis is explicit and realized by an infinite sequence of finite processes that lends itself to incorporating into other constructions. In the definition we use an infinite two-player game that is played in a Banach space Y with a basis (y_j) for an outcome \mathcal{O} . On turn k the first player chooses a tail subspace $[(y_j)_{j \geq m_k}]$ and the second player picks a vector $x_k \in [(y_j)_{j \geq m_k}]$. The second player is said to have a winning strategy for an outcome \mathcal{O} if no matter how the first player chooses, the sequence $(x_k)_{k=1}^\infty$ satisfies \mathcal{O} . Note that since the first player can push the supports of x_k ’s arbitrarily far out, the resulting sequence will be (a tiny perturbation of) a block basis of (y_j) .

Definition 5. We say that a basis (x_n) of a Banach space X is *two-player subsequentially C -reproducible* if for any sequence of positive numbers $(\epsilon_k)_{k \in \mathbb{N}}$ and isomorphic embedding T of X into a Banach space Y with a basis (y_n) , there is a winning strategy for the second player in a two-player game in Y for picking a subsequence $(Tx_{n_k})_{k=1}^\infty$ and blocks $(w_k)_{k=1}^\infty$ of the basis (y_n) such that

- (1) $\|Tx_{n_k} - w_k\| < \epsilon_k$ for each $k \in \mathbb{N}$,
- (2) (x_{n_k}) is C -equivalent to (x_n) .

Recall that any $C(K)$ space with countable compact K is isomorphic to some $C(\alpha)$ space where the latter denotes the space $C[1, \alpha]$ of continuous functions on a successor ordinal $\alpha + 1 < \omega_1$ equipped with the order topology, [7]. For a given compact metric space K , the corresponding α is determined as follows. Let

$$K^{(1)} = \{k : \exists k_n \in K, n = 1, 2, 3, \dots, k_n \neq k_m, m \neq n, k_n \rightarrow k\}$$

be the set of limit points of K . Put $K^{(\alpha+1)} = (K^{(\alpha)})^{(1)}$, and $K^{(\beta)} = \bigcap_{\alpha < \beta} K^{(\alpha)}$ if β is a limit ordinal. Let $o(K)$ be the smallest ordinal such that $K^{(o(K))}$ has finite cardinality or, if $K^{(\alpha)}$ is always infinite, let $o(K) = \omega_1$. If $\omega^\alpha \leq o(K) < \omega^{\alpha+1}$, then $C(K)$ is isomorphic to $C(\omega^{\omega^\alpha})$ [2].

The standard bases $(x_n^\alpha)_{n=0}^\infty$ of $C(\omega^\alpha)$ are described inductively. For $C(\omega)$, let $x_0^1 = \mathbb{1}_{(0, \omega]}$ and $x_n^1 = \mathbb{1}_{\{n\}}$ for all $n < \omega$. If the basis $(x_n^\gamma)_n$ for $C(\omega^\gamma)$ is defined, then for each $k < \omega$ let $x_{k,n}^\gamma$ have support in $(\omega^\gamma(k-1), \omega^\gamma k]$ and satisfy

$$(2.1) \quad x_{k,n}^\gamma(\rho) = x_n^\gamma(\rho - \omega^\gamma(k-1)) \text{ for } \omega^\gamma(k-1) < \rho \leq \omega^\gamma k.$$

Let $x_0^{\gamma+1} = \mathbb{1}_{(0, \omega^{\gamma+1}]}$, and $(x_j^{\gamma+1})_{j \geq 1}$ be an ordering of $\{x_{k,n}^\gamma : n = 0, 1, 2, \dots, k \in \mathbb{N}\}$ such that

$$(2.2) \quad \text{if } x_j^{\gamma+1} = x_{k,n}^\gamma \text{ and } x_m^{\gamma+1} = x_{k',n'}^\gamma \text{ and } n < n', \text{ then } j < m.$$

That is, the order of the basis is such that whenever the support of one function is contained in another, the top function precedes in the order. If γ is a limit ordinal, we fix a strictly increasing sequence (γ_k) with limit γ , and let $x_{k,n}^\gamma$ have support in $(\omega^{\gamma_{k-1}}, \omega^{\gamma_k}]$ and satisfy

$$x_{k,n}^\gamma(\rho) = x_n^{\gamma_k}(\rho - \omega^{\gamma_{k-1}}), \text{ for } \omega^{\gamma_{k-1}} < \rho \leq \omega^{\gamma_k}, k \in \mathbb{N}, n = 0, 1, \dots$$

where we set $\omega^{\gamma_0} = 0$. Then proceed analogously to define $(x_n^\gamma)_{n=0}^\infty$ where $x_0^\gamma = \mathbb{1}_{(0, \omega^\gamma]}$.

For $C_0(\omega^\alpha)$ a standard basis is $(x_n^\alpha)_{n=1}^\infty$. It is not hard to see with this construction that for any γ and n the sequence $(x_j^\gamma)_{j \in M}$ where

$M = \{j : \text{supp } x_j^\gamma \subsetneq \text{supp } x_n^\gamma\}$ is 1-equivalent to a standard basis of $C_0(\omega^\beta)$ for some $\beta < \gamma$. Also we have the following.

Fact 6. Consider the set of the supports of basis functions endowed with the partial order of inclusion. Let $M \subset \mathbb{N}$ and $\phi : M \rightarrow \mathbb{N}$ be injective. Then two subsequences $(x_i^\alpha)_{i \in M}$ and $(x_{\phi(i)}^\alpha)_{i \in M}$ are 1-equivalent if and only if ϕ induces an order isomorphism from $\{\text{supp } x_i^\alpha : i \in M\}$ to $\{\text{supp } x_{\phi(i)}^\alpha : i \in M\}$.

Note also that the basis is dependent on the sequence (γ_k) and the choices in the ordering of the $(x_{k,n})$, $k \in \mathbb{N}$. In this paper we are usually able to pass to a subsequence when needed. The following lemma shows that these choices of (γ_k) and ordering are a minor technical annoyance.

Lemma 7. *Suppose that (x_n) and (y_n) are standard bases of $C(\omega^\alpha)$, respectively, $C_0(\omega^\alpha)$, chosen as above. Then there exists a subsequence of (x_n) which is 1-equivalent to (y_n) and has closed span which is contractively complemented in $C(\omega^\alpha)$, $C_0(\omega^\alpha)$, respectively. Moreover the subsequence can be chosen by a two-player game.*

Proof. First observe that if we choose a subsequence of (x_n) so that if x_j is not in the subsequence then for all n such that the support of x_n is contained in the support of x_j , x_n is not in the subsequence, then the closed span of the subsequence is contractively complemented. Thus in the construction below we will choose a subsequence with this property.

The proof is by induction on the countable ordinals $\alpha \geq 1$. Because in the case of $C(\omega^\alpha)$ the procedure for choosing a basis requires that $x_1 = y_1 = \mathbb{1}_{(0, \omega^\alpha]}$, we only need to consider the case $C_0(\omega^\alpha)$. Our inductive hypothesis is that there is a strategy for the second player in a two player game such that for any $\beta \leq \alpha$ and subsequence $(x_n)_{n \in M}$ which is 1-equivalent to standard basis of $C_0(\alpha)$ and $(y_{p_k})_{k \in \mathbb{N}}$ which is 1-equivalent to a standard basis of $C_0(\beta)$, the second player is able to choose a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of $(x_n)_{n \in M}$ which is 1-equivalent to $(y_{p_k})_{k \in \mathbb{N}}$. The game requires that at turn k the first player presents a natural number l_k , $m_{k-1} < l_k$ and the second player must choose an element n_k of M so that $l_k < n_k$.

If $\alpha = 1$, the inductive hypothesis is clearly valid. Suppose that it holds for all $\alpha < \gamma$ and that $(x_n)_{n \in M}$ and $(y_{p_n})_{n \in \mathbb{N}}$ are subsequences of some standard bases of $C_0(\omega^\xi)$ for some $\xi \geq \gamma$, 1-equivalent to standard bases of $C_0(\omega^\gamma)$ and $C_0(\omega^\beta)$, respectively, for some $\beta \leq \gamma$. Let $(y_{q(0,l)})_{l \in \mathbb{N}}$ be the subsequence of $(y_{p_n})_{n \in \mathbb{N}}$ of elements of maximal support. For each $l \in \mathbb{N}$ let $(y_{q(l,k)})_{k \in \mathbb{N}}$ be the subsequence of $(y_{p_n})_{n \in \mathbb{N}}$ of all elements

y_{p_j} such that $p_j > q(0, l)$ and the support of y_{p_j} is contained in the support of $y_{q(0, l)}$.

The game begins with the choice of some natural number $l(1)$ by the first player. Note that $q(0, 1) = p_1$. $(y_{q(1, k)})_{k \in \mathbb{N}}$ is 1-equivalent to a standard basis of $C_0(\omega^{\gamma(1)})$ for some $\gamma(1) < \beta \leq \gamma$. Because $(x_n)_{n \in M}$ is 1-equivalent to a standard bases of $C_0(\omega^\gamma)$, there are infinitely many elements of maximal support $x_{m(i)}$, $i \in \mathbb{N}$, $m(i) \in M$, and for each i the elements x_n with $n > m(i)$ and support contained in the support of $x_{m(i)}$ is 1-equivalent to a standard basis of $C_0(\omega^{\beta(i)})$ where $\beta(i) < \gamma$ and either $\sup_i \beta(i) = \gamma$ or $\beta(i) + 1 = \gamma$ for infinitely many i . The second player in game chooses some i_1 and $n_1 = m(i_1)$ such that $\beta(i_1) \geq \gamma(1)$ and $m(i_1) > l(0, 1) = l(1)$. Let $(x_n)_{n \in M(1)}$, $M(1) \subset M$ be the subsequence of elements with support strictly contained in the support of x_{n_1} .

For the second turn of the game player 1 chooses an integer $l(2) > n_1$, and there are two possibilities: either $p_2 = q(0, 2)$ or $p_2 = q(1, 1)$. If $p_2 = q(0, 2)$, then $(y_{q(2, k)})_{k \in \mathbb{N}}$ is 1-equivalent to a standard basis of $C_0(\omega^{\gamma(2)})$ for some $\gamma(2) < \beta \leq \gamma$. The second player chooses $n_2 = m(i_2) > l(2)$ for some i_2 satisfying $\beta(i_2) \geq \gamma(2)$. Let $(x_n)_{n \in M(2)}$, $M(2) \subset M$, be the subsequence of elements with support strictly contained in the support of x_{n_2} . If $p_2 = p(q(1, 1))$ then game 1 is started with $(x_n)_{n \in M(1)}$ and $(y_{q(1, k)})_{k \in \mathbb{N}}$ and the integer $l(2) = l(1, 1) \geq n_1$. The second player in game 1 chooses $n_2 \in M(2)$ with $n_2 > l(1, 1)$ by the strategy from the inductive hypothesis for $\alpha = \beta(i_1)$ and $\beta = \gamma(1)$.

Proceeding in this fashion at each turn k either $p_k = q(0, j)$ for some j and the second player picks $n_k = m(i_j) > l(k) > n_{k-1}$ with $\beta(i_j) \geq \gamma(j)$ or $p_k = q(j, r)$ for some, j, r , and the r turn of the game j is played with $l(k) = l(j, r) > n_{k-1}$ to pick $n_k \in M(j)$. It is easy to see that the resulting sequence $(x_{n_k})_{k \in \mathbb{N}}$ is 1-equivalent to $(y_{p_n})_{n \in \mathbb{N}}$, and the closed span is contractively complemented. That completes the proof of the induction step and consequently, the lemma. \square

The subsequence $(x_n^\alpha)_{n=1}^\infty$ spans the subspace $C_0(\omega^\alpha)$ of $C(\omega^\alpha)$ of functions vanishing at ω^α . It is not hard to see that the biorthogonal functionals $(x_n^{\alpha*})_n$ are differences of point-mass measures, $\delta_{\gamma(n)} - \delta_{\gamma'(n)}$, where $\gamma(n)$ is the largest ordinal in the support of x_n^α and $\gamma'(n) = \gamma(k)$, where k is the largest integer strictly smaller than n such that the support of x_k^α contains the support of x_n^α . (For the top level basis elements let $\gamma'(n) = \omega^\alpha$.) It follows that for each m , $[x_n^{\alpha*} : n \leq m]$ is the span of point-mass measures and isometric to l_1^m , and therefore

$(x_n^\alpha)_{n=1}^\infty$ is a shrinking basis. Recall that this means that for every functional f on $C_0(\omega^\alpha)$, $\|f|_{[x_n^\alpha]_{n \geq k}}\| \rightarrow 0$ as $k \rightarrow \infty$.

The proof of Lemma 7 shows that the sequence $(x_n^\alpha)_{n=1}^\infty$ has many subsequences that are 1-equivalent to $(x_n^\alpha)_{n=1}^\infty$. In fact we can say even more.

Proposition 8. *A standard basis $(x_n^\alpha)_{n=1}^\infty$ of $C_0(\omega^\alpha)$ is two-player sub-sequentially 1-reproducible.*

Proof. For $\alpha = 1$, $(x_n^\alpha)_{n=1}^\infty$ is the unit vector basis of c_0 . Thus the result is immediate by a sliding hump argument and the fact that c_0 -basis is 1-subsymmetric (i.e., 1-equivalent to each of its subsequences). Suppose that for all $\beta < \alpha$, $(x_i^\beta)_i$ satisfies the conclusion. If $\alpha = \beta + 1$, then by definition of the basis (see 2.1) $(x_i^\alpha)_{i=1}^\infty$ consists of the sequence of top functions $(x_{k,0}^\beta)_{k=1}^\infty$ and under each a sequence $(x_{k,i}^\beta)_{i=1}^\infty$ which is equivalent to a standard basis of $C_0(\omega^\beta)$. By Lemma 7 any subsequence of $(x_i^\alpha)_i$ which consist of an infinite subsequence $(x_{k_j,0}^\beta)_j$ of top functions and $(x_{k_j,i}^\beta)_i$ bases has a subsequence which is 1-equivalent to $(x_i^\alpha)_i$. Thus the second player has a winning strategy in a block subspace game by alternating two winning strategies in a prescribed basis order; by picking an appropriate subsequence $(x_{k_j,0}^\beta)_j$ of top functions using the strategy for $\alpha = 1$ and for each k_j using the strategy from the inductive assumption to choose a subsequence of the corresponding $(x_{k_j,i}^\beta)_i$ bases.

The strategy in the limit ordinal case is similar since if α is the limit of the sequence (γ_k) , then the sequence of top functions $(x_0^{\gamma_k})_k$ is 1-equivalent to c_0 -basis and the basis elements below each are equivalent to a standard basis of $C_0(\omega^{\alpha_k})$ for some $\alpha_k < \alpha$. Again employing the strategy for $\alpha = 1$ on the top functions and from the inductive assumption, by following the strategy for α_{k_j} for those top functions chosen, in the required order, and using Lemma 7, the subsequence equivalent to the given standard basis of $C_0(\omega^\alpha)$ can be produced. \square

We will prove a much stronger statement taking advantage of *the weak injectivity property* of $C(K)$ spaces due to Pelczynski [9]: If a separable Banach space contains a subspace Y that is isomorphic to a $C(K)$ space, then there is a further subspace Z of Y such that Z is isomorphic to $C(K)$ and Z is *complemented* in X . In the case K is countable, we will show that Z can be realized as a subspace spanned by a subsequence of the reproducible basis and in fact, the second player has a winning strategy to produce such a subsequence. Thus we introduce the following general terminology.

Definition 9. We say that a basis (x_n) of a Banach space X is *two-player D -complementably subsequentially C -reproducible* if the second player has a winning strategy in the following modified two-player game. Let $(\epsilon_k)_{k \in \{0\} \cup \mathbb{N}}$ be a sequence of positive numbers, T be an isomorphic embedding of X into a Banach space Y with a basis (y_i) . Suppose that $l_0 = 0$, $G_0 = \emptyset$, $l_0 < l_1 < \dots < l_{k-1}$, a finite index set G_{k-1} , positive numbers $(\delta_n^{k-1})_{n \in G_{k-1}}$, and $(v_n^{k-1})_{n \in G_{k-1}} \subset Y^*$ have been chosen. On the k th turn, the first player chooses an integer i_k , a finite number of elements $(u_n^k)_{n \in F_k}$ of Y^* , positive real numbers $(\rho_n^k)_{n \in F_k}$ and also chooses a finite set of blocks $(b_j)_{j \in J_k} \subset [y_i : l_{k-1} < i \leq i_k]$ satisfying $|v_l^{k-1}(b)| < \delta_l^{k-1} \|b\|$ for all $b \in [b_j : j \in J_k]$ and $l \in G_{k-1}$. The second player chooses an integer $l_k > i_k$, a finite set $M_k \subset \mathbb{N}$ with $M_{k-1} < M_k$, i.e., $\max M_{k-1} < \min M_k$, a finite set of blocks $(w_m^k)_{m \in M_k} \subset [y_i : i_k < i \leq l_k]$ with $\sum_{m \in M_k} \|Tx_m - w_m^k\| < \epsilon_k$, and with $|u_n^k(x)| < \rho_n^k \|x\|$ for all $x \in [Tx_m : m \in M_k]$ and $n \in F_k$, and chooses a finite number of elements $(v_n^k)_{n \in G_k}$ of Y^* and positive real numbers $(\delta_n^k)_{n \in G_k}$. The second player wins if for $M = \bigcup_{k=1}^{\infty} M_k$,

- (1) $(x_m)_{m \in M}$ is C -equivalent to the basis $(x_n)_{n=1}^{\infty}$,
- (2) there is a projection P of norm at most $\|T\| \|T^{-1}\| D$ from $[\{b_j : j \in J_k, k \in \mathbb{N}\} \cup \{Tx_m : m \in M\}]$ onto $[Tx_m : m \in M]$ with $\|Pz\| \leq \epsilon_0 \|z\|$ for all $z \in [b_j : j \in J_k, k \in \mathbb{N}]$.

For p in a countable compact space K , δ_p denotes the Dirac evaluation functional on $C(K)$. (We use δ_k^n for small positive numbers below but the indices make the distinction clear.) Note that $\delta_{\omega^\alpha} = 0$ for the case $C_0(\omega^\alpha)$ below. The next lemma is a recasting of the core of Pelczynski's result for the countable case [9].

Lemma 10. *Let $\alpha < \omega_1$, and let S be an isomorphic embedding of $C(\omega^\alpha)$ (respectively, of $C_0(\omega^\alpha)$) into a separable Banach space Y , let (x_n^α) be a standard basis of $C(\omega^\alpha)$ (respectively, of $C_0(\omega^\alpha)$), and let $(y_\rho^*)_{\rho \leq \omega^\alpha} \subset 2\|(S^*)^{-1}\|B_{Y^*}$ satisfy $S^*y_\rho^* = \delta_\rho$ for all $\rho \leq \omega^\alpha$. Then there is a compact subset Γ of $[1, \omega^\alpha]$ homeomorphic to $[1, \omega^\alpha]$ and a (weak*) compact subset $(w_\rho^*)_{\rho \in \Gamma}$ of Y^* such that*

(i) $S^*w_\rho^* = \delta_\rho$ for all $\rho \in \Gamma$, for each isolated point γ of Γ is an isolated point of $[1, \omega^\alpha]$, $w_\gamma^* = y_\gamma^*$, and the map $\rho \rightarrow w_\rho^*$ is a homeomorphism,

(ii) there is a subsequence of (x_n^α) equivalent to (x_n^α) , with contractively complemented closed linear span such that the restriction to Γ induces an isomorphism R from the span of the subsequence onto $C(\Gamma)$ (respectively, $C_0(\Gamma)$), and $R^*\delta_\rho = S^*w_\rho^*$ for all $\rho \in \Gamma$.

Proof. The proof is by induction on α . Suppose $\alpha = 1$. By passing to a subsequence $(y_k^*)_{k \in K}$, we have that $y_k^* \rightarrow y^*$ in the weak* topology for some $y^* \in Y^*$. Let $\Gamma = K \cup \{\omega\}$ and put $w_\omega^* = y^*$. The desired subsequence of the basis $(\mathbb{1}_n)_{n \in \mathbb{N}}$ is $(\mathbb{1}_k)_{k \in K}$ in the case $C_0(\omega)$, and $x_0^1 = \mathbb{1}_{[1, \omega]}$ followed by $(\mathbb{1}_k)_{k \in K}$ for the case $C(\omega)$. All of the requirements are clearly satisfied.

Now suppose that the lemma holds for all $\beta < \alpha$. Because $(x_n^\alpha)_{n=1}^\infty$ is a standard basis of $C_0(\omega^\alpha)$, the supports A_k , $k \in K$, of the top level elements in the basis, $(x_k^\alpha)_{k \in K}$, are intervals homeomorphic to $[1, \gamma_k]$ where (γ_k) is of one of two types. If $\alpha = \beta + 1$ for some ordinal β , $\gamma_k = \omega^\beta$, else there exist $\alpha_k \nearrow \alpha$ and $\gamma_k = \omega^{\alpha_k}$ for all k . For each k , the elements of the basis which are supported strictly inside A_k , $(x_{k,j})$, are a standard basis of $C_0(\gamma_k)$. By induction, for each k we can find subsets Γ_k of A_k and $(w_\rho^*)_{\rho \in \Gamma_k}$ and subsequences $(x_{k,j})_{j \in J_k}$ of the bases of $C_0(A_k)$ as in the conclusion. For each k let ζ_k denote the highest order point in Γ_k , i.e., $\Gamma_k^{(\xi)} = \{\zeta_k\}$ for some $\xi < \omega_1$. By passing to a subsequence we may assume that $(w_{\zeta_k}^*)_{k \in K}$ converges to some y^* . For each k there is a w^* -neighborhood \mathcal{N}_k of $w_{\zeta_k}^*$ such that if $y_k^* \in \mathcal{N}_k$, then $(y_k^*)_{k \in K}$ converges to y^* . For each k by replacing Γ_k by a slightly smaller set and $(x_{k,j})_{j \in J_k}$ by a corresponding subsequence $(x_{k,j})_{j \in J'_k}$ we may assume that $w_\rho^* \in \mathcal{N}_k$ for all $\rho \in \Gamma_k$. Let $\Gamma = \{\omega^\alpha\} \cup \bigcup_{k \in K} \Gamma_k$ and $w_{\omega^\alpha}^* = y^*$. In the case of $C(\omega^\alpha)$ a subsequence of the basis equivalent to a standard basis is $\mathbb{1}_{[1, \omega^\alpha]}$ followed by the elements of the bases $(x_k)_{k \in K}$, $(x_{k,j})_{j \in J_k}$, $k \in K$ in the required order. Some further thinning using Lemma 7 may be required to get a subsequence equivalent to (x_n) . In the case $C_0(\omega^\alpha)$ we omit $\mathbb{1}_{[1, \omega^\alpha]}$. The required properties are easily verified. \square

Proposition 11. *Let α be a countable ordinal. A standard basis $(x_n^\alpha)_{n=1}^\infty$ of $C_0(\omega^\alpha)$ is two-player 2-complementably subsequentially 1-reproducible.*

Proof. We will use induction on α with an inductive hypothesis which will be described after the first step is proved. Let T be an isomorphism of $C_0(\omega^\alpha)$ into a Banach space Y with a bimonotone basis $(y_n)_{n=1}^\infty$.

Assume $\alpha = 1$ and let $(\epsilon_k)_{k=0}^\infty$ be a sequence of positive numbers and $\epsilon'_0 = \epsilon_0/(2\|T\|)$. For each $n \in \mathbb{N}$, let w_n^* be a Hahn-Banach extension of $(T^{-1})^*\delta_n$ from $T(C_0(\omega^\alpha))$ to Y . By passing to a subsequence and restricting to an isometric subspace as given by Lemma 10, we may assume that (w_n^*) converges weak* to some $w^* \in Y^*$. Note that w^* is 0 on $T(C_0(\omega^\alpha))$. Let $d_n^* = w_n^* - w^*$ for each n . For use in later steps of the induction we will not make full use of the fact that $d_0^* = w^* \lim d_n^* = 0$. Instead we will use that it is small on certain elements. More precisely, we put $\delta_0^0 = \epsilon'_0/2^2$ and consider d_0^* as an element chosen before the first

player's first turn which imposes the condition on that player's choice of $(b_j)_{j \in J_1}$, $|v_0^0(b)| = |d_0^*(b)| < \delta_0^0 \|b\| = (\epsilon'_0/2^2) \|b\|$ for all $b \in [b_j : j \in J_1]$. Because (x_n) is a two-player subsequentially 1-reproducible basis, we can use the strategy for that game and impose additional requirements. We now describe the second player's move in the game.

Suppose that at step k the first player has chosen an integer i_k , a finite set of blocks $(b_j)_{j \in J_k} \subset [y_n : n \leq i_k]$ of the basis (y_n) satisfying $|v_l^{k-1}(b)| < \delta_l^{k-1} \|b\|$ for all $b \in [b_j : j \in J_k]$, $l \in G_{k-1}$, and has chosen elements $(u_n^k)_{n \in F_k}$ of Y^* and positive real numbers $(\rho_n^k)_{n \in F_k}$. Because (d_n^*) converges to d_0^* , $|d_0^*(b)| < \delta_0^{l-1} \|b\|$ for all $b \in [b_j : j \in J_l]$, $l \leq k$, and (Tx_n) is weakly null, there exists m'_k so that if $m \geq m'_k$, $|d_m^*(b)| < |d_0^*(b)| + (\delta_0^{l-1}/2) \|b\|$ for all $b \in [b_j : j \in J_l]$, $l \leq k$, and $|u_n^k(Tx_m)| < \rho_n^k \|Tx_m\|$ for all $n \in F_k$. By the strategy for the sequential reproducibility game the second player chooses $m_k \geq m'_k$ and l_k such that there is some block $w_{m_k} \in [y_n : i_k < n \leq l_k]$ such that $\|Tx_{m_k} - w_{m_k}\| < \epsilon_k$. The second player sets $(v_l^k)_{l \in G_k} = (d_{m_l}^*)_{l=0}^k$ and $(\delta_l^k)_{l \in G_k} = (\epsilon'_0/2^{2(k+1)})_{l=0}^k$ for the conditions on the first player's next turn.

Because the second player uses the strategy from the reproducibility game $(x_{m_k})_{k \in \mathbb{N}}$ is 1-equivalent to (x_n) . Let $Pz = \sum_{k=1}^{\infty} d_{m_k}^*(z) Tx_{m_k}$ for all $z \in Y$. Observe that for all k, k' , $d_{m_k}^*(Tx_{m_{k'}}) = \delta_{m_k}(\mathbb{1}_{m_{k'}})$. Hence $PTx_{m_k} = Tx_{m_k}$. If $z = \sum_{k=1}^K \sum_{j \in J_k} a_{k,j} b_j^k$, then

$$\begin{aligned} |d_{m_{k'}}^*(z)| &= |d_{m_{k'}}^*(\sum_{k=1}^{k'} \sum_{j \in J_k} a_{k,j} b_j^k) + d_{m_{k'}}^*(\sum_{k=k'+1}^K \sum_{j \in J_k} a_{k,j} b_j^k)| \\ &\leq \sum_{k=1}^{k'} \left(|d_0^*(\sum_{j \in J_k} a_{k,j} b_j^k)| + (\delta_0^{k-1}/2) \|\sum_{j \in J_k} a_{k,j} b_j^k\| \right) + \sum_{k=k'+1}^K \delta_0^{k-1} \|\sum_{j \in J_k} a_{k,j} b_j^k\| \\ &\leq |d_0^*(z)| + \sum_{k=1}^{k'} (\epsilon'_0/2^{2k}) \|z\| + \sum_{k=k'+1}^K \epsilon'_0/2^{2k} \|z\| < |d_0^*(z)| + (\epsilon'_0) \|z\|. \end{aligned}$$

This completes the first step of the induction.

For the induction hypothesis we actually want more than the statement of the proposition. This is because for $\alpha > 1$, the projection formula (from the weak injectivity property of Pelczynski) is more involved. Namely, we require that we are able to choose pairs of elements, x_m from the basis and $d_m^* \in Y^*$, so that $T^*d_m^* = \delta_{\gamma(m)} - \delta_{\omega^\alpha} = \delta_{\gamma(m)}$, the natural mapping $S : [x_m : m \in M] \rightarrow C_0(\Gamma)$ where $\Gamma = \{\gamma(m) : m \in M\}$ satisfying $(Sx_m)(\gamma(k)) = x_m(\gamma(k))$ is a surjective isometry, and the projection P is of the form TEV where $V : Y \rightarrow C_0(\Gamma)$ is

defined by $(Vz)(\gamma(m)) = d_m^*(z)$ for all $z \in Y$, E is the extension operator which maps $C_0(\Gamma)$ into $C_0(\omega^\alpha)$ with range in $[x_m : m \in M]$ with $SE = I$. Explicitly

$$Ef = \sum_{m \in M} (f(\gamma(m)) - \sum_{\substack{m' \in M: \\ m' \neq m, \\ x_{m'} \geq x_m}} f(\gamma(m'))) x_m$$

or equivalently,

$$E_f(\beta) = \begin{cases} f(\beta) & \text{if } \beta \in \Gamma, \\ f(\gamma(m)) & \text{if } x_m(\beta) = 1, x_{m'}(\beta) = 0 \text{ for all } m' > m, \\ 0 & \text{else.} \end{cases}$$

Notice that the norm of the projection P is at most $\|T\| \sup_{m \in M} \|d_m^*\|$.

Now suppose that the following induction hypothesis holds for all $\beta < \alpha$ and $\alpha > 1$.

For all sequences of positive numbers $(\epsilon_k)_{k=0}^\infty$, maps $T : C_0(\omega^\beta) \rightarrow Y$, standard bases $(x_n)_{n \in \mathbb{N}}$, with extensions $(d_n^*)_{n \in \{0\} \cup \mathbb{N}}$ of $((T^{-1})^* \delta_{\gamma(n)})_{n \in \{0\} \cup \mathbb{N}}$, as in Lemma 10, there is a winning strategy for the second player in the complementably sequentially reproducible basis game to produce $(x_m)_{m \in M}$ and $(d_m^*)_{m \in M}$ as in the game which also satisfy these properties.

- (1) The elements of M are chosen one by one, i.e., at each of the second player's turns only one m and w_m are chosen with $\|Tx_m - w_m\|$ as small as desired. (The required bound on the perturbation does not need to be known until the step at which m and w_m are chosen.)
- (2) For each $k \in \{0\} \cup \mathbb{N}$, $(v_n^k)_{n \in G_k} = (d_m^*)_{m \in M_k}$ where M_k is the set with elements $\{0\}$ and the first k elements of M .
- (3) The projection is of the form described above.
- (4) For all $z \in [b_j : j \in \cup_k J_k]$, $|d_m^*(z)| < |d_0^*(z)| + (\epsilon_0/(2\|T\|))\|z\|$.

Here d_0^* is an extension of $(T^{-1})^* \delta_{\omega^\beta}$.

Let $T : C_0(\omega^\alpha) \rightarrow Y$, $(x_n)_{n \in \mathbb{N}}$ be a standard basis of $C_0(\omega^\alpha)$ with extensions $(f_n^*)_{n \in \{0\} \cup \mathbb{N}}$ of $((T^{-1})^* \delta_{\gamma(n)})_{n \in \{0\} \cup \mathbb{N}}$. First we apply Lemma 10 to replace the original map T by its restriction to a subspace isometric to $C_0(\omega^\alpha)$ spanned by a subsequence of the basis equivalent to it such that for each $\gamma \leq \omega^\alpha$, we have w_γ^* , a Hahn-Banach extension of $(T^{-1})^*(\delta_\gamma)$, such that $\delta_\gamma \rightarrow w_\gamma^*$ is a w^* -homeomorphism. In this way we may assume that the original T has these properties.

The given basis of $C_0(\omega^\alpha)$ contains a subsequence $(x_n)_{n \in N_0}$ such that for every $k \notin N_0$, there is some $n \in N_0$ with $x_k \leq x_n$ (pointwise) and if $n, m \in N_0$, then $x_n \cdot x_m = 0$. The support of each x_n , $n \in N_0$, is

homeomorphic to $[1, \omega^{\zeta_n}]$ for some $\zeta_n < \alpha$. To simplify the situation choose a sequence $(n(i))_{i=1}^\infty$ in N_0 of distinct elements such that $\zeta_{n(i)} \nearrow \alpha$, if α is a limit ordinal, and $\zeta_{n(i)} = \beta$, if $\beta + 1 = \alpha$ for some ordinal β . We may discard all x_n such that $x_n \cdot x_{n(i)} = 0$ for all i . Because we can apply Lemma 7 at the end of the argument to obtain a subsequence equivalent to the original standard basis of $C_0(\omega^\alpha)$, it will be sufficient to produce a subsequence of the basis equivalent to *some* standard basis of $C_0(\omega^\alpha)$ that satisfies all of the other requirements.

For each $i \in \mathbb{N}$ let $N_i = \{m : x_m \leq x_{n(i)}, m \neq n(i)\}$. $(x_m)_{m \in N_i}$ is a standard basis for $C_0(\omega^{\gamma_{n(i)}})$ for some $\gamma_{n(i)} < \alpha$, and $(d_m^*)_{m \in \{n(i)\} \cup N_i} = (w_{\gamma(m)}^* - w_{\omega^\alpha}^*)_{m \in \{n(i)\} \cup N_i}$ is a corresponding sequence of extensions of the inverse images of the Dirac measures. Thus for each i the induction hypothesis applies. $(x_m)_{m \in N_0}$ is a standard basis for $C_0(\omega)$ and $(d_m^*)_{m \in \{0\} \cup N_0} = (w_{\gamma(m)}^* - w_{\omega^\alpha}^*)_{m \in \{0\} \cup N_0}$, where $\gamma(0) = \omega^\alpha$, is a corresponding sequence of extensions of the inverse images of the Dirac measures. Therefore there is a winning strategy as in the case $\beta = 1$. The remainder of the argument is interweaving all of the strategies to produce the required strategy for α . Below at each step of the induction there will be a finite but increasing number of games employed so we will number the games as they arise as game $0, 1, 2, \dots$. We will include an extra subscript when needed to indicate parameters associated with a particular game. For example $\delta_{j,n}^k$ would be associated with game j . Parameters for the combined game α will have a single subscript.

Let the sequence of positive numbers $(\epsilon_k)_{k=0}^\infty$ be given and let $\delta_0^0 = \epsilon_0/(4\|T\|)$. We may assume that (ϵ_k) is non-increasing. The first player chooses an integer i_1 , a finite set of blocks $(b_j)_{j \in J_1}$ in $[y_k : k \leq i_1]$, such that $|d_0^*(b)| < \delta_0^0 \|b\|$ for all $b \in [b_j : j \in J_1]$, $(u_n^1)_{n \in F_1}$ from the dual of Y and positive real numbers $(\rho_n^1)_{n \in F_1}$. The second player views this as the first move of game 0 and uses the strategy for $\beta = 1$ with $\epsilon_{0,0} = \epsilon_0/2^2$ to choose $l_{0,1} > i_1$, $m_{0,1} \in N_0$, and a block $w_{m_{0,1}} \in [y_i : i_1 < i \leq l_{0,1}]$ such that $\|Tx_{m_{0,1}} - w_{m_{0,1}}\| < \epsilon_{0,1} = \epsilon_1$ and $|u_n^1(Tx_{m_{0,1}})| < \rho_n^1 \|Tx_{m_{0,1}}\|$ for all $n \in F_1$. Then the second player chooses functionals $(v_{0,n}^1)_{n \in G_{0,1}}$ and positive numbers $(\delta_{0,n}^1)_{n \in G_{0,1}}$ for the first player's next turn of game 0. We set $l_1 = l_{0,1}$, $m_1 = m_{0,1}$, $w_{m_1} = w_{m_{0,1}}$, $(v_n^1)_{n \in G_1} = (v_{0,n}^1)_{n \in G_{0,1}}$, and $(\delta_n^1)_{n \in G_1} = (\delta_{0,n}^1)_{n \in G_{0,1}}$ for game α .

The first player chooses an integer $i_2 > l_1$, $(u_n^2)_{n \in F_2}$ from Y^* , positive real numbers $(\rho_n^2)_{n \in F_2}$, and blocks $(b_j)_{j \in J_2} \subset [y_i : l_1 < i \leq i_2]$ satisfying $|v_n^1(b)| < \delta_n^1 \|b\|$ for all $b \in [b_j : j \in J_2]$, $n \in G_1$. The second player uses the next move of the strategy for $\beta = 1$ to choose $l_{0,2} > i_2$, $m_{0,2} \in N_0$, and a block $w_{m_{0,2}} \in [y_i : i_2 < i \leq l_{0,2}]$ such that $\|Tx_{m_{0,2}} - w_{m_{0,2}}\| < \epsilon_{0,2} = \epsilon_2$ and $|u_n^2(Tx_{m_{0,2}})| < \rho_n^2 \|Tx_{m_{0,2}}\|$ for all $n \in F_2$. Then the second

player chooses functionals $(v_{0,n}^2)_{n \in G_{0,2}}$ and positive numbers $(\delta_{0,n}^2)_{n \in G_{0,2}}$ for the first player's next turn of game 0. We set $l_2 = l_{0,2}$, $m_2 = m_{0,2}$, $w_{m_2} = w_{m_{0,2}}$, $(v_n^2)_{n \in G_2} = (v_{0,n}^2)_{n \in G_{0,2}}$, and $(\delta_n^2)_{n \in G_2} = (\delta_{0,n}^2/2)_{n \in G_{0,2}}$ for game α .

For the third turn of the game α , the first player chooses an integer $i_3 > l_2$, $(u_n^3)_{n \in F_3}$ from Y^* , positive real numbers $(\rho_n^3)_{n \in F_3}$, and blocks $(b_j)_{j \in J_3} \subset [y_i : l_2 < i \leq i_3]$ satisfying $|v_n^2(b)| < \delta_n^2 \|b\|$ for all $b \in [b_j : j \in J_3]$, $n \in G_2$. This time the second player considers this his first move of game 1 with $\epsilon_{1,0} = \epsilon_0/2^4$, the basis $(x_n)_{n \in N'_1}$ where $N'_1 = N_{m_{0,1}}$ and $(d_{1,m}^*)_{m=0}^\infty = (d_m^*)_{m \in \{m_{0,1}\} \cup N'_1} = (w_m^* - w_{\omega^\alpha}^*)_{m \in \{m_{0,1}\} \cup N'_1}$, $(d_{1,0}^* = d_{m_{0,1}}^*)$ and uses the strategy for $\beta_1 = \zeta_{m_{0,1}}$ with conditions $i_{1,1} = i_3$, $(u_{1,n}^1)_{n \in F_{1,1}} = (u_n^3)_{n \in F_3}$, $(\rho_{1,n}^1)_{n \in F_{1,1}} = (\rho_n^3)_{n \in F_3}$, and $(b_j)_{j \in J_{1,1}} = (b_j)_{j \in J_1 \cup J_2 \cup J_3}$. The second player chooses $l_{1,1} > i_3$, $m_{1,1} \in N'_1$, and a block $w_{m_{1,1}} \in [y_i : i_3 < i \leq l_{1,1}]$ such that $\|Tx_{m_{1,1}} - w_{m_{1,1}}\| < \epsilon_{1,1} = \epsilon_3$ and $|u_n^3(Tx_{m_{1,1}})| < \rho_n^3 \|Tx_{m_{1,1}}\|$ for all $n \in F_3$. The second player also chooses functionals $(v_{1,n}^1)_{n \in G_{1,1}}$ and positive numbers $(\delta_{1,n}^1)_{n \in G_{1,1}}$ for the first player's next turn of game 1. We set $l_3 = l_{1,1}$, $m_3 = m_{1,1}$, $w_{m_3} = w_{m_{1,1}}$, $(v_n^3)_{n \in G_3} = (v_{1,n}^1)_{n \in G_{1,1}} \uplus (v_{0,n}^2)_{n \in G_{0,2}}$ and $(\delta_n^3)_{n \in G_3} = (\delta_{1,n}^1)_{n \in G_{1,1}} \uplus (\delta_{0,n}^2)_{n \in G_{0,2}}$ for game α , where \uplus denotes concatenation of the finite sequences.

Now we briefly describe how to continue. The key point is to include in the conditions for each move the conditions imposed by all of the moves of the games in progress. In order to write this more precisely we need to introduce some notation. The moves are taken in Cantor order $(0, 1), (0, 2), (1, 1), (0, 3), \dots$ for the elements of $(\{0\} \cup \mathbb{N}) \times \mathbb{N}$. The turn k of the game α is considered by the second player to be the move $\text{mv}(k)$ of game $\text{gm}(k)$ where

$$k = \text{tn}(\text{gm}(k), \text{mv}(k)) = 1 + \text{gm}(k) + \sum_{r=0}^{\text{mv}(k) + \text{gm}(k) - 1} r,$$

$$\text{gm}(k) \geq 0, \text{ and } \text{mv}(k) \geq 1.$$

The first player makes the move for turn k by choosing an integer i_k , a finite set of blocks $(b_j)_{j \in J_k}$ in $[y_i : l_{k-1} < i \leq i_k]$, such that $|v_n^{k-1}(b)| < \delta_n^{k-1} \|b\|$ for all $b \in [b_j : j \in J_k]$, $n \in G_{k-1}$, $(u_n^k)_{n \in F_k}$ from the dual of Y and positive real numbers $(\rho_n^k)_{n \in F_k}$. Let $k' = \text{gm}(k)$ and $k'' = \text{mv}(k)$. The second player considers this his move k'' of game k' with $\epsilon_{k',0} = \epsilon_0/(2^{k'+1} \|T\|)$, the basis $(x_n)_{n \in N'_{k'}}$ where $N'_{k'} = N_{m_{0,k'}}$ and $(d_{k',m}^*)_{m=0}^\infty = (d_m^*)_{m \in \{m_{0,k'}\} \cup N'_{k'}} = (w_m^* - w_{\omega^\alpha}^*)_{m \in \{m_{0,k'}\} \cup N'_{k'}}$, $(d_{k',0}^* = d_{m_{0,k'}}^*)$ and uses the strategy for $\beta_{k'} = \zeta_{m_{0,k'}}$ with conditions

$i_{k',k''} = i_k$, $(u_{k',n}^k)_{n \in F_{k',k''}} = (u_n^k)_{n \in F_k}$, $(\rho_{k',n}^k)_{n \in F_{k',k''}} = (\rho_n^k)_{n \in F_k}$, and $(b_j)_{j \in J_{k',k''}} = (b_j)_{j \in J_{k-k'-k''+1} \cup J_{k-k'-k''+2} \cup \dots \cup J_k}$, if $k'' > 1$, $(b_j)_{j \in J_{k',k''}} = (b_j)_{j \in J_1 \cup J_2 \cup \dots \cup J_{k-1}}$, if $k'' = 1$. The second player chooses $l_{k',k''} > i_{k',k''} = i_k$, $m_{k',k''} \in N'_{k'}$, and a block $w_{m_{k',k''}} \in [y_i : i_k < i \leq l_{k',k''}]$ such that $\|Tx_{m_{k',k''}} - w_{m_{k',k''}}\| < \epsilon_{k',k''} = \epsilon_k$ and $|u_n^k(Tx_{m_{k',k''}})| < \rho_n^k \|Tx_{m_{k',k''}}\|$ for all $n \in F_k$. The second player also chooses functionals $(v_{k',n}^{k''})_{n \in G_{k',k''}}$ and positive numbers $(\delta_{k',n}^{k''})_{n \in G_{k',k''}}$ for the first player's next turn of game k' . For turn $k+1$ of game α , we set $l_k = l_{k',k''}$, $m_k = m_{k',k''}$, $w_{m_k} = w_{m_{k',k''}}$,

$$(v_n^k)_{n \in G_k} = \biguplus_{\substack{0 \leq \kappa' \leq k', \\ \kappa'' = k' + k'' - \kappa'}} (v_{\kappa',n}^{\kappa''})_{n \in G_{\kappa',\kappa''}} \biguplus_{\substack{k' < \kappa' \leq k' + k'', \\ \kappa'' = k' + k'' - \kappa' - 1}} (v_{\kappa',n}^{\kappa''})_{n \in G_{\kappa',\kappa''}}$$

and

$$(\delta_n^k)_{n \in G_k} = \biguplus_{\substack{0 \leq \kappa' \leq k', \\ \kappa'' = k' + k'' - \kappa'}} (\delta_{\kappa',n}^{\kappa''} / 2^{k' - \kappa' + 1})_{n \in G_{\kappa',\kappa''}} \biguplus_{\substack{k' < \kappa' \leq k' + k'', \\ \kappa'' = k' + k'' - \kappa' - 1}} (\delta_{\kappa',n}^{\kappa''} / 2^{k'' + 1 + \kappa''})_{n \in G_{\kappa',\kappa''}}$$

Observe that $(v_n^k)_{n \in G_k}$ is in fact $(d_m^*)_{m \in \{0\} \cup \{m_i : i \leq k\}}$. This completes turn k .

Let $M = \{m_k : k \in \mathbb{N}\} = \bigcup_{j=0}^{\infty} M'_j$, where $M'_j = \{m_{j,i} : i \in \mathbb{N}\}$, and for each $j \in \mathbb{N} \cup \{0\}$. It is easy to see that because for each j , $(x_{m(j,i)})_{i \in \mathbb{N}}$ was constructed by using the inductive hypothesis, the order we have used produces a standard basis of $C_0(\omega^\alpha)$ with one basis element chosen at each turn. Also we chose m_k such that $\|Tx_{m_k} - w_{m_k}\| < \epsilon_k$. We define the projection P onto $[Tx_m : m \in M]$ by TEV where V is the evaluation at $\{d_m^* : m \in M\} \cup \{0\}$, which is homeomorphic to $[1, \omega^\alpha]$ and can be identified with $\Gamma = \{\gamma(m) : m \in M\} \cup \{\omega^\alpha\}$, and E is the extension map from $C_0(\Gamma)$ onto $[x_m : m \in M] \subset C_0(\omega^\alpha)$. The norm of P is at most $2\|T\|\|T^{-1}\|$. Let $z \in [b_j : j \in J_k, k \in \mathbb{N}]$. Fix k and observe that m_k was chosen by the strategy for game $k' = \text{gm}(k)$. Therefore

for $\text{gm}(k) > 0$,

$$\begin{aligned} z \in [b_j : j \in J_{k',i}, i \in \mathbb{N}] \\ = [\{b_j : j \in J_{k-k'-i+l}, 1 \leq l \leq k' + i, i = 2, 3, \dots\} \\ \cup \{b_j : j \in J_l, 1 \leq l \leq k\}] = [b_j : j \in J_i, i \in \mathbb{N}], \end{aligned}$$

$$\begin{aligned} |d_{m_k}^*(z)| &= |d_{m(k', \text{mv}(k))}^*(z)| < |d_{m(0,k')}^*(z)| + (\epsilon_{k',0}/(2\|T\|))\|z\| \\ &< |d_0^*(z)| + (\epsilon_{0,0}/(2\|T\|))\|z\| + (\epsilon_0/(2^{2k'+1}\|T\|))\|z\| \\ &= |d_0^*(z)| + (\epsilon_0/(2\|T\|))(4^{-1} + 4^{-k'})\|z\|. \end{aligned}$$

If $k' = 0$, we have the simpler estimate

$$\begin{aligned} |d_{m_k}^*(z)| &= |d_{m(0, \text{mv}(k))}^*(z)| < |d_0^*(z)| + (\epsilon_{0,0}/(2\|T\|))\|z\| \\ &= |d_0^*(z)| + (\epsilon_0/(2\|T\|))4^{-1}\|z\|. \end{aligned}$$

This shows that the induction hypothesis is satisfied. Notice that if we pass to a subsequence of $(x_m)_{m \in M}$ that is equivalent to $(x_n)_{n \in \mathbb{N}}$ by using Lemma 7 we have the same conditions satisfied. It follows from the estimates above that for $z \in [b_j : j \in J_i, i \in \mathbb{N}]$, $\|Pz\| \leq \|T\|(\epsilon_0/(2\|T\|))(4^{-1} + 4^{-k'})\|z\| < (\epsilon_0/2)\|z\|$. Therefore a standard basis of $C_0(\omega^\alpha)$ is 2-complementably subsequentially 1-reproducible. \square

We can use the interweaving approach from the argument used in the proof of the previous proposition to prove the following.

Corollary 12. *Suppose that for each n , Z_n is a Banach space with a 2-player D -complementably subsequentially C -reproducible basis $(z_{n,k})$. Let Y be a Banach space with a basis and for each n let $T_n : Z_n \rightarrow Y$ be an isomorphism such that $\|T_n^{-1}\| \leq 1$ and $\sup \|T_n\| = K < \infty$. Then for every $\epsilon > 0$ for each n there is a subsequence $(z_{n,k})_{k \in M_n}$ of $(z_{n,k})$ such that $(z_{n,k})_{k \in M_n}$ is C -equivalent to the basis of Z_n , $(T_n z_{n,k})_{k \in M_n}$ is a perturbation of a block of the basis of Y , disjointly supported from the blocks for $(T_m z_{m,k})$, all $m \neq n$, and there is a projection P_n , $\|P_n\| \leq KD$, from $[T_m z_{m,k} : m \in \mathbb{N}, k \in M_m]$ onto $[T_n z_{n,k} : k \in M_n]$ such that for any $z \in [T_m z_{m,k} : m \in \mathbb{N}, k \in M_m, m \neq n]$, $\|Pz\| < \epsilon\|z\|$. Moreover the sequences $(z_{n,k})_{k \in M_n}$ are produced by a 2-player game.*

Proof. (Sketch) Let $\epsilon > 0$, and let $\epsilon_k = \epsilon/(K2^{2k+2})$, for all $k \in \mathbb{N}$. Without loss of generality we may assume that the basis of Y is bi-monotone. Let game 0 be the overall game with players 1 and 2. For each $n \in \mathbb{N}$ we will have a game n whose second player is following the strategy to produce the required subsequence of $(z_{n,k})_{k \in \mathbb{N}}$. As in the

previous proof we will use the first subscript on parameters to denote the game.

We begin with player 1 as the first player in game 0. Player 1 chooses $i_{0,1}$, a finite sequence $(u_{0,n}^1)_{n \in F_{0,1}} \subset Y^*$, positive numbers $(\rho_{0,n}^1)_{n \in F_{0,1}}$, and a finite set of blocks $(b_{0,j})_{j \in J_{0,1}} \subset [y_i : i \leq i_{0,1}]$. Player 2 views this as the move of the first player in turn 1 of game 1 with $\epsilon_{i,k} = \epsilon_1/2^{2(k+1)}$, for $k \in \{0\} \cup \mathbb{N}$, $i_{1,1} = i_{0,1}$, $(u_{1,n}^1)_{n \in F_{1,1}} = (u_{0,n}^1)_{n \in F_{0,1}}$, $(\rho_{1,n}^1)_{n \in F_{1,1}} = (\rho_{0,n}^1)_{n \in F_{0,1}}$, and $(b_{1,j})_{j \in J_{1,1}} = (b_{0,j})_{j \in J_{0,1}}$. Using the strategy to produce a subsequence of $(z_{1,k})_{k \in \mathbb{N}}$ player 2 chooses an integer $l_{1,1}$, a finite set $M_{1,1} \subset \mathbb{N}$, a finite set of blocks $(w_{1,m}^1)_{m \in M_{1,1}} \subset [y_i : i_{1,1} < i \leq l_{1,1}]$ with $\sum_{m \in M_{1,1}} \|T_1 z_{1,m} - w_{1,m}\| < \epsilon_1$, and with $|u_{1,n}^1(x)| < \rho_{1,n}^1 \|x\|$ for all $x \in [T_1 x_m : m \in M_{1,1}]$, a finite sequence $(v_{1,n}^1)_{n \in G_{1,1}} \subset Y^*$, and positive numbers $(\delta_{1,n}^1)_{n \in G_{1,1}}$. Let $l_{0,1} = l_{1,1}$, $(v_{0,n}^1)_{n \in G_{0,1}} = (v_{1,n}^1)_{n \in G_{1,1}}$, and $(\delta_{0,n}^1)_{n \in G_{0,1}} = (\delta_{1,n}^1)_{n \in G_{1,1}}$.

For the second turn of game 0, Player 1 chooses $i_{0,2}$, a finite sequence $(u_{0,n}^2)_{n \in F_{0,2}} \subset Y^*$, positive numbers $(\rho_{0,n}^2)_{n \in F_{0,2}}$, and a finite set of blocks $(b_{0,j})_{j \in J_{0,2}} \subset [y_i : l_{0,1} < i \leq i_{0,2}]$, such that $|v_n^1(b)| < \delta_n^1 \|b\|$ for all $b \in [b_{0,j} : j \in J_{0,2}]$, $n \in G_{1,1}$. Player 2 views this as the move of the first player in turn 2 of game 1, i.e., $i_{1,2} = i_{0,2}$, $(u_{1,n}^2)_{n \in F_{1,2}} = (u_{0,n}^2)_{n \in F_{0,2}}$, $(\rho_{1,n}^2)_{n \in F_{1,2}} = (\rho_{0,n}^2/2)_{n \in F_{0,2}}$, and $(b_{1,j})_{j \in J_{1,2}} = (b_{0,j})_{j \in J_{0,2}}$. Player 2 chooses an integer $l_{1,2}$, a finite set $M_{1,2} \subset \mathbb{N}$, a finite set of blocks $(w_{1,m}^2)_{m \in M_{1,2}} \subset [y_i : i_{1,2} < i \leq l_{1,2}]$ with $\sum_{m \in M_{1,2}} \|T_1 z_{1,m} - w_{1,m}\| < \epsilon_{1,2}$, and with $|u_{1,n}^2(x)| < \rho_{1,n}^2 \|x\|$ for all $x \in [T_1 x_m : m \in M_{1,2}]$, a finite sequence $(v_{1,n}^2)_{n \in G_{1,2}} \subset Y^*$, and positive numbers $(\delta_{1,n}^2)_{n \in G_{1,2}}$. Let $l_{0,2} = l_{1,2}$, $(v_{0,n}^2)_{n \in G_{0,2}} = (v_{1,n}^2)_{n \in G_{1,2}}$, and $(\delta_{0,n}^2)_{n \in G_{0,2}} = (\delta_{1,n}^2/2)_{n \in G_{1,2}}$.

For the third turn of game 0 Player 1 chooses $i_{0,3}$, a finite sequence $(u_{0,n}^3)_{n \in F_{0,3}} \subset Y^*$, positive numbers $(\rho_{0,n}^3)_{n \in F_{0,3}}$, and a finite set of blocks $(b_{0,j})_{j \in J_{0,3}} \subset [y_i : l_{0,2} < i \leq i_{0,3}]$, such that $|v_n^2(b)| < \delta_n^2 \|x\|$ for all $b \in [b_{0,j} : j \in J_{0,3}]$, $n \in G_{1,2}$.

Player 2 now begins game 2 by setting $\epsilon_{2,k} = \epsilon_2/2^{2(k+1)}$, $k \in \{0\} \cup \mathbb{N}$, $i_{2,1} = i_{1,3}$, $(b_j)_{j \in J_{2,1}} = (b_j)_{j \in J_{1,1}} \uplus (w_{1,m}^1)_{m \in M_{1,1}} \uplus (b_j)_{j \in J_{1,2}} \uplus (w_{1,m}^2)_{m \in M_{1,2}} \uplus (b_{0,j})_{j \in J_{0,3}}$, $(u_{2,n}^1)_{n \in F_{2,1}} = (u_{0,n}^3)_{n \in F_{0,3}} \uplus (v_{1,n}^2)_{n \in G_{1,2}}$, and $(\rho_{2,n}^1)_{n \in F_{2,1}} = (\rho_{0,n}^3)_{n \in F_{0,3}} \uplus (\delta_{1,n}^2/2)_{n \in G_{1,2}}$. Using the strategy for $(z_{2,n})_{n \in \mathbb{N}}$, Player 2 chooses an integer $l_{2,1}$, a finite set $M_{2,1} \subset \mathbb{N}$, a finite set of blocks $(w_{2,m}^k)_{m \in M_{2,1}} \subset [y_i : i \leq l_{2,1}]$ with $\sum_{m \in M_{2,1}} \|T_2 z_{2,m} - w_{2,m}\| < \epsilon_{2,1}$, and with $|u_{2,n}^1(x)| < \rho_{2,n}^1 \|x\|$ for all $x \in [T_2 z_{2,m} : m \in M_{2,1}]$, a finite sequence $(v_{2,n}^1)_{n \in G_{2,1}} \subset Y^*$, and positive numbers $(\delta_{2,n}^1)_{n \in G_{2,1}}$.

Continuing in this way player 2 works through the next turns of the games in progress and then starts the next new game. Player 2

adjusts the parameters so that in each round of turns the estimates sum properly. \square

By using a modification of the projection from the previous result we can position the sequences $(T_m z_{m,k})_{k \in K_m}$, $m \in \mathbb{N}$, $m \neq n$, in the kernel of a projection onto $[T_n z_{n,k} : k \in K_n]$. We believe that the next lemma is well-known and is due to Pelczynski but we do not have a precise reference.

Lemma 13. *Suppose that X is a Banach space, X_1 and Z are subspaces of X , $0 < \epsilon < 1$, and P is a projection from x onto X_1 and $\|Pz\| \leq \epsilon\|z\|$, for all $z \in Z$. Then there is a projection Q from $X_1 + Z$ onto X_1 such that $Qz = 0$ for all $z \in Z$.*

Proof. Let $R = (I - P)|_Z$ and let $W = (I - P)Z$. Because $\epsilon < 1$, $\|Rz\| \geq (1 - \epsilon)\|z\|$ for all $z \in Z$. Thus R is an isomorphism from Z onto W . Let $Q = I - R^{-1}(I - P)$. $Qx = x$ for all $x \in X_1$ and $Qz = 0$ for all $z \in Z$. \square

3. AN ORDINAL INDEX FOR UNCONDITIONAL SUMS

An essential technical tool that we need is an ordinal index that is closely related the basis index introduced by Bourgain. The new index, however, will be defined for an unconditional sum of Banach spaces rather than for one dimensional subspaces (i.e., a basis).

Recall that given a set A , a tree \mathcal{T} on A is a partially ordered family of finite tuples of elements of A such that if $(a_1, a_2, \dots, a_n) \in \mathcal{T}$ then $(a_1, a_2, \dots, a_{n-1}) \in \mathcal{T}$. The partial order is by extension so that $(a_1, a_2, \dots, a_n) \leq (b_1, b_2, \dots, b_k)$ iff $n \leq k$ and $a_j = b_j$ for $j = 1, 2, \dots, n$. A branch of \mathcal{T} is a maximal totally ordered subset. The tuples in the tree are called nodes, the element a_j of the node (a_1, a_2, \dots, a_n) will be the j th entry, and if there is a minimal node r that is comparable to all nodes, we will say that the tree is rooted at r .

A derivation of \mathcal{T} is defined by deleting nodes with no proper extensions. Let $\mathcal{T}^{(1)} = \{b \in \mathcal{T} : \exists c \in \mathcal{T}, b \leq c, b \neq c\}$. If $\mathcal{T}^{(\alpha)}$ has been defined, let $\mathcal{T}^{(\alpha+1)} = (\mathcal{T}^{(\alpha)})^{(1)}$. If β is a limit ordinal, $\mathcal{T}^{(\beta)} = \bigcap_{\alpha < \beta} \mathcal{T}^{(\alpha)}$. If α is the smallest countable ordinal such that $\mathcal{T}^{(\alpha)} = \emptyset$, then the order of the tree is set as $o(\mathcal{T}) = \alpha$. Otherwise we put $o(\mathcal{T}) = \omega_1$. If A is a separable complete metric space and \mathcal{T} on A is closed and has order ω_1 , then T has an infinite branch.

Suppose that A is a separable Banach space and (y_i) is a normalized basis of a Banach space Y . If $K < \infty$ and the nodes are finite normalized sequences $(a_i)_{i=1}^n$ in A that are K -equivalent to the initial segment $(y_i)_{i=1}^n$, the index gives a very useful tool for checking whether

the Banach space Y embeds into the Banach space A . Indeed, for this Bourgain basis tree one only needs to check that the index is ω_1 . Unfortunately this tree does not seem to be suitable for handling sums of spaces, so we need to make some major modifications.

Let Z be a Banach space with a 1-unconditional basis (z_n) and for each n let Y_n be a Banach space with norm $\|\cdot\|_n$. Then by $(\sum Y_n)_Z$ we denote the direct sum of Y_n 's with respect to (z_n) . That is, the space is the Banach space of sequences $(y_n)_{n=1}^\infty$, $y_n \in Y_n$ for all n , with finite norm $\|(y_n)\|_Z = \|\sum_{n=1}^\infty \|y_n\|_n z_n\|_Z$. Observe that this Z -sum is well-behaved with respect to uniformly bounded sequences of operators acting on the coordinate spaces.

Let Z and Y_n , $n \in \mathbb{N}$, be as above and, in addition, fix constants $C, D > 0$ and a Banach space X . Consider a tree \mathcal{T} of tuples consisting of pairs of subspaces and isomorphisms

$$((X_1, T_1), (X_2, T_2), \dots, (X_k, T_k)),$$

where X_j is a subspace of X and T_j is an isomorphism from X_j onto Y_j such that $\|T_j\| \leq C$, $\|T_j^{-1}\| \leq 1$, and for all $x_j \in X_j$, we have

$$\left\| \sum_{j=1}^k x_j \right\| \leq \|(T_j x_j)\|_Z \leq D \left\| \sum_{j=1}^k x_j \right\|, \quad 1 \leq j \leq k.$$

We partially order \mathcal{T} by extension and the order of the tree is defined as before. We call \mathcal{T} a $(\sum Y_n)_Z$ -tree in X with constants C, D , and the order of the tree is referred as $(\sum Y_n)_Z$ index.

Even if we assume that all of the spaces are separable, we cannot proceed as before to establish that trees with index ω_1 have an infinite branch. We do not know whether this is even true. However we are able to prove the following result which is satisfactory for our purposes. The proof actually shows that there is an infinite branch which in a sense close to branches of the given tree.

Theorem 14. *Let Z be a Banach space with a normalized 1-unconditional basis (z_n) , and let X and Y_n , $n \in \mathbb{N}$ be separable Banach spaces. If \mathcal{T} is a $(\sum Y_n)_Z$ -tree in X with index ω_1 and constants C, D , then X contains a subspace which is D -isomorphic to $(\sum Y_n)_Z$.*

Proof. Let W be a Banach space with a basis $(w_k)_{k=1}^\infty$ that contains X . For each $j, k \in \mathbb{N}$ choose a subset $\Omega_{j,k}$ of $[w_i : i \leq j] \cap B_W$, where B_W denotes the unit ball of W , such that for all $w \in [w_i : i \leq j] \cap B_W$, there exists $w' \in \Omega_{j,k}$ such that $\|w - w'\| < 2^{-k}$. For each $n, m \in \mathbb{N}$ let $\Upsilon_{n,m}$ be a finite subset of B_{Y_n} such that $\Upsilon_{n,m} \subsetneq \Upsilon_{n,m+1}$ for all m and $\bigcup_{m=1}^\infty \Upsilon_{n,m}$ is dense in B_{Y_n} .

Let the branches of \mathcal{T} be indexed by A so that for each $\alpha \in A$, $((X_{\alpha,1}, T_{\alpha,1}), \dots, (X_{\alpha,k}, T_{\alpha,k}))$, $1 \leq k < M_\alpha$, $M_\alpha \leq \infty$, is a branch and if $M_\alpha < \infty$, then $((X_{\alpha,1}, T_{\alpha,1}), \dots, (X_{\alpha,M_\alpha-1}, T_{\alpha,M_\alpha-1}))$ is a terminal node. For each $(X_{\alpha,1}, T_{\alpha,1})$, the initial node of the branch $\alpha \in A$, find $n = n(\alpha, 1, 1)$ such that for each $y \in \Upsilon_{1,1}$ there exists $w(y) \in \Omega_{n,2}$ such that $\|w(y) - T_{\alpha,1}^{-1}y\| < 2^{-1}$ and w is one-to-one. We may assume that if $(X_{\alpha,1}, T_{\alpha,1}) = (X_{\gamma,1}, T_{\gamma,1})$ then $n(\alpha, 1, 1) = n(\gamma, 1, 1)$. For each α let $W(\alpha, 1, 1) = \{w(y) : y \in \Upsilon_{1,1}\}$. For each $y \in \Upsilon_{1,1}$ there may be more than one possibility for $w(y)$ but we choose one to put in the set and use the same choice for all α with the same initial node. ($w(\alpha, 1, 1)(y)$ actually depends on α and the index $(1, 1)$ but we are suppressing this in the notation for now.) There are countably many possibilities for integers $n(\alpha, 1, 1)$, sets $W(\alpha, 1, 1)$, and bijections $w(\alpha, 1, 1) : \Upsilon_{1,1} \rightarrow W(\alpha, 1, 1)$, so there must be an integer $n(1, 1)$, a bijection $w(1, 1)$ and a subset $W(1, 1)$ of $\Omega_{n(1,1),2}$ such that if

$$A_1 = \{\alpha : n(\alpha, 1, 1) = n(1, 1) \text{ and } W(\alpha, 1, 1) = W(1, 1), \\ w(\alpha, 1, 1)(y) = w(1, 1)(y) \text{ for all } y \in \Upsilon_{1,1}\},$$

the subtree \mathcal{T}_1 of nodes with first entry $(X_{\alpha,1}, T_{\alpha,1})$ for $\alpha \in A_1$ has index ω_1 . This completes the first step of an induction.

Next for each $((X_{\alpha,1}, T_{\alpha,1}), (X_{\alpha,2}, T_{\alpha,2}))$, which occurs as the second node of a branch $\alpha \in A_1$, we find integers $n(\alpha, 1, 2)$ and $n(\alpha, 2, 1)$ and subsets $W(\alpha, 2, 1)$ of $\Omega_{n(\alpha,2,1),2}$ and $W(\alpha, 1, 2)$ of $\Omega_{n(\alpha,1,2),3}$ such that for each $y_2 \in \Upsilon_{2,1}$, $y_1 \in \Upsilon_{1,2}$, there are $w(y_2) \in W(\alpha, 2, 1)$, $w(y_1) \in W(\alpha, 1, 2)$ such that $\|T_{\alpha,2}^{-1}(y_2) - w(y_2)\| < 2^{-1}$ and $\|T_{\alpha,1}^{-1}(y_1) - w(y_1)\| < 2^{-2}$. Here we again assume that $w(\cdot)$ is a bijection on each set and that the integers $n(\alpha, 1, 2)$ and $n(\alpha, 2, 1)$ and $w(\cdot)$ depend only on the second node in the branch, $((X_{\alpha,1}, T_{\alpha,1}), (X_{\alpha,2}, T_{\alpha,2}))$. As before there are countably many choices for the integers, bijections and finite sets, so there are integers $n(2, 1)$, $n(1, 2)$, bijections $w(2, 1)$ and $w(1, 2)$, and subsets $W(2, 1)$ of $\Omega_{n(2,1),2}$ and $W(1, 2)$ of $\Omega_{n(1,2),3}$ such that if

$$A_2 = \left\{ \alpha \in A_1 : n(\alpha, 1, 2) = n(1, 2), n(\alpha, 2, 1) = n(2, 1), \right. \\ W(\alpha, 1, 2) = W(1, 2), w(\alpha, 1, 2)(y) = w(1, 2)(y) \text{ for all } y \in \Upsilon_{1,2}, \\ \left. W(\alpha, 2, 1) = W(2, 1), w(\alpha, 2, 1)(y) = w(2, 1)(y) \text{ for all } y \in \Upsilon_{2,1} \right\},$$

then the tree \mathcal{T}_2 of nodes from the branches in A_2 has index ω_1 .

Continuing in this way we find a decreasing sequence of subsets $(A_m)_{m=1}^\infty$ of A , positive integers $n(i, j)$, and subsets $W(i, j)$ of $\Omega_{n(i,j),j+1}$, $i, j \in \mathbb{N}$, such that if $i + j \leq m + 1$, there is a bijection $w(i, j)$ from

$\Upsilon_{i,j}$ onto $W(i,j)$ with $\|T_{\alpha,i}^{-1}(y) - w(i,j)(y)\| < 2^{-j}$ for all $y \in \Upsilon_{i,j}$. Moreover, the subtree \mathcal{T}_m of nodes from branches in A_m has order ω_1 .

Because $\Upsilon_{i,j} \subset \Upsilon_{i,k}$, all $k > j$, for each $i \in \mathbb{N}$, we can define

$$B_i = \left\{ x : \text{there exists } y \in \Upsilon_{i,j}, j \in \mathbb{N}, \lim_k w(i,k)(y) = x \right\}.$$

(The limit is in the norm of W .) Notice that for $x \in B_i$, for each $\alpha \in A_m, j \leq k \leq m+1-i$, if $x_{\alpha,i} = T_{\alpha,i}^{-1}(y)$,

$$\|x_{\alpha,i} - w(i,k)(y)\| = \|T_{\alpha,i}^{-1}(y) - w(i,k)(y)\| < 2^{-k}.$$

Thus for any choice $\alpha(m) \in A_m, m > i, \lim_{m \rightarrow \infty} x_{\alpha(m),i} = x \in X$, and $\|x\| \leq 1$. Further $\|T_{\alpha,i}^{-1}(y)\| \geq C^{-1}\|y\|$ so that $\|x\| \geq C^{-1}\|y\|$.

If $y \in \Upsilon_{i,j}$ for some j , then for all $\alpha \in A_m, \|T_{\alpha,i}^{-1}(y) - w(i,k)(y)\| < 2^{-k}$, for $j \leq k \leq m+1-i$. Thus $\|w(i,k)(y) - w(i,k')(y)\| \leq 2^{-\min(k,k')+1}$ if $j \leq k' \leq m+1-i$ also. Thus we may define a map $T_i^{-1} : \bigcup_j \Upsilon_{i,j} \rightarrow B_i$ by $T_i^{-1}(y) = \lim_k w(i,k)(y)$. On its domain T_i^{-1} is the pointwise limit of uniformly bounded linear maps, thus T_i^{-1} continuously extends to B_{Y_i} as an affine map and to all of Y_i by scaling. Because it is bounded below, it is an isomorphism. Let X_i be the range of T_i^{-1} .

It remains to verify that $[x : x \in X_i, i \in \mathbb{N}]$ is isomorphic to $(\sum_i Y_i)_Z$. This follows from the fact that for any finite sequence $(x_i)_{i=1}^k, x_i \in B_{X_i}$ and $\epsilon > 0$, we can find j such that for each $i \leq k$ there exists $y_i \in \Upsilon_{i,j}$ such that

$$\|T_i^{-1}(y_i) - x_i\| < \epsilon k^{-1}.$$

For m sufficiently large if $\alpha \in A_m$ then

$$\|T_i^{-1}(y_i) - T_{\alpha,i}^{-1}(y_i)\| < \epsilon k^{-1}$$

for $1 \leq i \leq k$. Thus

$$\left\| \sum_{i=1}^k x_i - \sum_{i=1}^k T_{\alpha,i}^{-1}(y_i) \right\| < 2\epsilon$$

and, because we have assumed (z_i) is normalized,

$$\begin{aligned} (3.1) \quad \left\| \sum_{i=1}^k \|T_i x_i\| z_i - \sum_{i=1}^k \|y_i\| z_i \right\| &\leq \sum_{i=1}^k \|T_i x_i - y_i\| \\ &\leq C \sum_{i=1}^k \|x_i - T_i^{-1} y_i\| < 2C\epsilon. \end{aligned}$$

We have that for $\alpha \in A_m$, $m > j + k + 1$,

$$D^{-1} \left\| \sum_{i=1}^k \|y_i\| z_i \right\| \leq \left\| \sum_{i=1}^k T_\alpha^{-1} y_i \right\| \leq \left\| \sum_{i=1}^k \|y_i\| z_i \right\|$$

and consequently

$$\begin{aligned} D^{-1} \left\| \sum_{i=1}^k \|y_i\| z_i \right\| - 2\epsilon &\leq \left\| \sum_{i=1}^k T_\alpha^{-1} y_i \right\| - 2\epsilon \leq \left\| \sum_{i=1}^k x_i \right\| \leq \\ &\left\| \sum_{i=1}^k T_\alpha^{-1} y_i \right\| + 2\epsilon \leq \left\| \sum_{i=1}^k \|y_i\| z_i \right\| + 2\epsilon. \end{aligned}$$

Using the estimate (3.1) We obtain

$$\begin{aligned} D^{-1} \left\| \sum_{i=1}^k \|T_i x_i\| z_i \right\| - 2C\epsilon D^{-1} - 2\epsilon \\ \leq \left\| \sum_{i=1}^k x_i \right\| \leq \left\| \sum_{i=1}^k \|T_i x_i\| z_i \right\| + 2C\epsilon + 2\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have

$$D^{-1} \left\| \sum_{i=1}^k \|T_i x_i\| z_i \right\| \leq \left\| \sum_{i=1}^k x_i \right\| \leq \left\| \sum_{i=1}^k \|T_i x_i\| z_i \right\|,$$

proving the result. \square

4. $C(K)$ SUBSPACES OF ELASTIC SPACES

In the previous sections we have developed the tools that will allow us to generalize the argument of Johnson and Odell and prove the following.

Proposition 15. *Let $K, C, D \geq 1$ be constants. Suppose that X is a separable K -elastic space and $(Y_n)_{n=1}^\infty$ is a sequence of spaces with two-player D complementably C -reproducible bases that embed into X . Then X contains a subspace isomorphic to $(\sum_{n=1}^\infty Y_n)_{c_0}$.*

Proof. We can assume that the elastic space X is contained in a Banach space W with a bi-monotone basis (w_n) . We will show by an inductive construction that for every $\epsilon > 0$ and each countable limit ordinal α , X contains a $(\sum_{n=1}^\infty Y_n)_{c_0}$ -tree of order at least α with both constants $K(1+\epsilon)$. To do this we will construct Banach spaces V^α , $\alpha < \omega_1$ which are isomorphic to subspaces of X and contain $(\sum_{n=N}^\infty Y_n)_{c_0}$ -trees of order α for each $N \geq 1$.

In order to avoid cluttering the arguments with multiplication by various values of the form $(1 + \delta)$ that are incurred from tiny perturbations of block bases, we will suppress these throughout.

For each $n \in \mathbb{N}$, let $(y_{n,k})_{k=1}^\infty$ be a two-player D complementably C -reproducible basis for Y_n , and let T_n be an isomorphism from Y_n into X with $\|T_n\| \leq K$ and $\|T_n^{-1}\| \leq 1$. Let $0 < \delta < 1$. By interweaving the two-player games (See Corollary 12.) we can find subsequences $(y_{n,k})_{k \in K_n}$ of $(y_{n,k})_{k=1}^\infty$ for all $n \in \mathbb{N}$, such that $(y_{n,k})_{k \in K_n}$ is equivalent to $(y_{n,k})_{k \in \mathbb{N}}$, for all $n \in \mathbb{N}$, $(T_n y_{n,k})_{k \in K_n, n \in \mathbb{N}}$ is (equivalent to) a block of (w_j) , and for each m there is a projection P_m from $V = [T_n y_{n,k} : k \in K_n, n \in \mathbb{N}]$ onto $[T_m y_{m,k} : k \in K_m]$ with $\|P_m y\| \leq \delta \|y\|$ for all $y \in [T_n y_{n,k} : k \in K_n, n \in \mathbb{N}, n \neq m]$. Let R_n be the basis equivalence from $(y_{n,k})_{k \in K_n}$ to $(y_{n,k})_{k=1}^\infty$. By Lemma 13, for each m we may assume that the projection P_m is zero on $[T_n y_{n,k} : k \in K_n, n \in \mathbb{N}, n \neq m]$.

For each $i \in \mathbb{N}$ define a norm $\|\cdot\|_i$ on V by

$$\|y\|_i = \sup \left\{ \|R_m T_m^{-1} P_m y\| : m \in \mathbb{N} \right\} \vee \frac{\|y\|}{iKC}.$$

Let $B = \sup_i \|R_i T_i^{-1} P_i\|$. Then $B \leq CD'$ where $D' = (1 + \delta)^{-1}(2 + D)$. (See the proof of Lemma 13.) Clearly $\frac{\|y\|}{iKC} \leq \|y\|_i \leq B\|y\|$ for all $y \in V$. Notice that if $y \in [T_j y_{j,k} : k \in K_j]$ for some j then $\|y\|_i = \|R_j T_j^{-1} y\|$. Thus $[T_j y_{j,k} : k \in K_j]$ with norm $\|\cdot\|_i$ is isometric to Y_j . For each $i \in \mathbb{N}$ let V^i be the space V with norm $\|\cdot\|_i$.

It is clear from the construction that if $F \subset \mathbb{N}$ with $|F| \leq i$, $v = \sum_{j \in F} v_j$ and $v_j \in [T_j y_{j,k} : k \in K_j]$, then $\|v\|_i = \max\{\|v_j\|_i : j \in F\}$. Indeed,

$$\begin{aligned} \|v\|_i &= \sup \left\{ \|R_m T_m^{-1} P_m v\| : m \in \mathbb{N} \right\} \vee \frac{\|v\|_X}{iKC} \\ &\leq \max \left\{ \|R_m T_m^{-1} v_j\| : j \in F \right\} \vee \frac{1}{iKC} \sum_{j \in F} \|v_j\|_X \\ &= \max\{\|v_j\|_i : j \in F\}. \end{aligned}$$

Conversely,

$$\begin{aligned} \|v\|_i &\geq \sup \left\{ \|R_m T_m^{-1} P_m v\| : m \in \mathbb{N} \right\} \geq \max \left\{ \|R_m T_m^{-1} v_j\| : j \in F \right\} \\ &\geq \max\{\|v_j\|_i : j \in F\}. \end{aligned}$$

This shows if $F = \{N, N+1, \dots, N+p\}$, $p < i$, $(([T_j y_{j,k} : k \in K_j], R_j T_j^{-1}))_{j \in F}$ is a node of a $(\sum_{n=N}^\infty Y_n)_{c_0}$ -tree.

Note that the definition of the norm on V^i produces a hereditary property. If for each $n \in \mathbb{N}$, we pass to a subsequence of $(y_{n,k})_{k \in K_n}$ which is equivalent, then the nodes in X formed from pairs with first

coordinate the image under T_n of the closed span of the subsequence, we again will have a node of a $(\sum_{n=N}^{\infty} Y_n)_{c_0}$ -tree.

Thus for all $N \geq 1$ there are branches of a $(\sum_{n=N}^{\infty} Y_n)_{c_0}$ -tree of length i in V^i . Since X is K -elastic, for each $i \in \mathbb{N}$ there is an isomorphism S_i from V^i into X such that $\|v\|_i \leq \|S_i v\| \leq K\|v\|_i$ for all $v \in V^i$. Thus X has the tree index (with constants K) at least ω .

We now extend this a little by observing that we can embed substantial parts of the spaces V^i simultaneously into X as disjointly supported blocks of (w_j) . Let $\delta > 0$ and let $(y_{i,n,k})_{k=1}^{\infty}$ be a two-player complementably sequentially reproducible basis of the subspace of V^i spanned by $(Ty_{n,k})_{k \in K_n}$ (in norm $\|\cdot\|_i$). By the definition of $\|\cdot\|_i$ this subspace is isometric to Y_n . By interweaving the two player games for $(y_{i,n,k})_{k=1}^{\infty}$, $i, n \in \mathbb{N}$, we can find subsequences $(y_{i,n,k})_{k \in K_{i,n}}$ of $(y_{i,n,k})_{k=1}^{\infty}$ for all $i, n \in \mathbb{N}$, such that $\{S_i y_{i,n,k} : k \in K_{i,n}, i, n \in \mathbb{N}\}$ is (equivalent to) a block of (w_j) in some order, and for each j, m there is a projection $P_{j,m}$ from $V^{\omega} := [S_i y_{i,n,k} : k \in K_{i,n}, i, n \in \mathbb{N}]$ onto $[S_j y_{j,m,k} : k \in K_{j,m}]$ with $\|P_{j,m} y\| \leq \delta \|y\|$ for all $y \in [S_i y_{i,n,k} : k \in K_{i,n}, i, n \in \mathbb{N}, (i, n) \neq (j, m)]$. Let $R_{i,n}$ be the basis equivalence from $(y_{i,n,k})_{k \in K_{i,n}}$ to $(y_{i,n,k})_{k=1}^{\infty}$. By Lemma 13, for all j, m we may assume that the projection $P_{j,m}$ is zero on $[S_i y_{i,n,k} : k \in K_n, i, n \in \mathbb{N}, (i, n) \neq (j, m)]$.

Define a norm on V^{ω} by

$$\|y\|_{\omega} = \sup \left\{ \|R_{j,m} S_j^{-1} P_{j,m} y\| : j, m \in \mathbb{N} \right\} \vee \frac{\|y\|}{KC},$$

where $B = \sup_{j,m} \|R_{j,m} S_j^{-1} P_{j,m}\|$. As before $\frac{\|y\|}{KC} \leq \|y\|_{\omega} \leq B\|y\|$ for all $y \in V^{\omega}$. For each i , if $|F| \leq i$ and for each $n \in F$, $x_{i,n} \in S_i[y_{i,n,k} : k \in K_{i,n}]$, then $\|\sum_{n \in F} x_{i,n}\|_{\omega} = \sup_{n \in F} \|R_{i,n} S_i^{-1} x_{i,n}\| = \|\sum_{n \in F} R_{i,n} S_i^{-1} x_{i,n}\|_i$. Consequently, the nodes of length less than or equal to i of the $(\sum_{n=N}^{\infty} Y_n)_{c_0}$ -tree of V^i are nodes of the $(\sum_{n=N}^{\infty} Y_n)_{c_0}$ -tree in the corresponding subspace of V^{ω} . In particular, V^{ω} contains a $(\sum_{n=N}^{\infty} Y_n)_{c_0}$ -tree of order ω for each $N \geq 1$.

Using the property that X is K elastic, we can find a subspace of X K -isomorphic to V^{ω} . Observe that V^{ω} has the same hereditary property with respect to equivalent subsequences of the $(y_{i,n,k})$ that V^i has. This completes the first part of the induction.

Now assume that for all limit ordinals $\beta < \alpha$ we have constructed spaces V^{β} containing a $(\sum_{n=N}^{\infty} Y_n)_{c_0}$ -tree of order β for each $N \geq 1$ and that V^{β} is isomorphic to a subspace of X . We assume that this tree property is preserved under passing to equivalent subsequences of the bases of each coordinate in each node. Further we can assume that the tree has only countably many nodes and hence that there are

countably many subspaces $Y_{n,m}$, $n, m \in \mathbb{N}$ such that $Y_{n,m}$ is isometric to Y_n for each $m \in \mathbb{N}$ and that each node of the tree is of the form

$$((Y_{N,m(N)}, J_{N,m(N)}), \dots, (Y_{N+k,m(N+k)}, J_{N+k,m(N+k)})).$$

If we need to refer to these subspaces for more than one β , then we will add a superscript β , e.g., $Y_{2,3}^\beta$. A two-player complementably sequentially reproducible basis for $Y_{n,m}$ will be denoted $(y_{n,m,k})_{k=1}^\infty$.

There are two cases to consider. If α is a limit ordinal of the form $\beta + \omega$, let T^ω be an isomorphism of V^ω into X and let T^β be an isomorphism of V^β into X as given by the elastic property. By interweaving the two-player games for the bases $(y_{n,m,k}^\gamma)_{k=1}^\infty$ for each $n, m \in \mathbb{N}$, and $\gamma = \omega, \beta$, we can find subsequences $(y_{n,m,k}^\omega)_{k \in K_{n,m}^\omega}$ of $(y_{n,m,k}^\omega)_{k=1}^\infty$, $(y_{n,m,k}^\beta)_{k \in K_{n,m}^\beta}$ of $(y_{n,m,k}^\beta)_{k=1}^\infty$, for all $n, m \in \mathbb{N}$, such that $\{T^\gamma y_{n,m,k}^\gamma : k \in K_{n,m}^\gamma, n, m \in \mathbb{N}, \gamma = \omega, \beta\}$ is (equivalent to) a block basis of (w_j) in some order, and for each $\eta \in \{\omega, \beta\}, i, j \in \mathbb{N}$ there is a projection $P_{i,j}^\eta$ from $V^{\beta+\omega} := [T^\gamma y_{n,m,k}^\gamma : k \in K_{n,m}^\gamma, n, m \in \mathbb{N}, \gamma = \omega, \beta]$ onto $[T^\eta y_{i,j,k}^\eta : k \in K_{i,j}^\eta]$ with $\|P_{i,j}^\eta y\| \leq \delta \|y\|$ for all $y \in [T^\gamma y_{n,m,k}^\gamma : k \in K_{n,m}^\gamma, n, m \in \mathbb{N}, \gamma \in \{\omega, \beta\}, (\gamma, n, m) \neq (\eta, i, j)]$. For each γ, n, m , let $R_{n,m}^\gamma$ be the basis equivalence from $(y_{n,m,k}^\gamma)_{k \in K_{n,m}^\gamma}$ to $(y_{n,m,k}^\gamma)_{k=1}^\infty$. By Lemma 13, for each i, j, η we may assume that the projection $P_{i,j}^\eta$ is zero on $[T^\gamma y_{n,m,k}^\gamma : k \in K_{n,m}^\gamma, n, m \in \mathbb{N}, \gamma \in \{\omega, \beta\}, (\gamma, n, m) \neq (\eta, i, j)]$.

By the inductive assumption for each $N, M \in \mathbb{N}$, V^β contains a $(\sum_{n=M+N}^\infty Y_n)_{c_0}$ -tree of order β with node entries drawn from the subspaces $Y_{n,m}^\beta$, $n, m \in \mathbb{N}$. Because $(y_{n,m,k}^\beta)_{k \in K_n}$ is C -equivalent to the given basis of $Y_{n,m}^\beta$ and the stability of $(\sum_{n=M+N}^{M+N+l} Y_n)_{c_0}$ under isomorphisms of the Y_n , X contains a $(\sum_{n=M+N}^\infty Y_n)_{c_0}$ -tree of order at least β with constants CKD and CK and node entries from $X_{n,m}^\beta = [T^\beta y_{n,m,k}^\beta : k \in K_{n,m}^\beta]$, $n, m \in \mathbb{N}$. Similarly, X contains a $(\sum_{n=M}^\infty Y_n)_{c_0}$ -tree of order at least ω with constants CKD and CK and the node entries from $X_{n,m}^\omega = [T^\omega y_{n,m,k}^\omega : k \in K_{n,m}^\omega]$, $n, m \in \mathbb{N}$. In particular, we can find a branch containing a node of length N ,

$$((X_{M,m(M)}^\omega, S_{M,m(M)}^\omega), \dots, (X_{M+N-1,m(M+N-1)}^\omega, S_{M+N-1,m(M+N-1)}^\omega))$$

where $S_{j,m(j)}^\omega = I_{j,m(j)}^\omega R_{j,m(j)}^\omega (T^\omega)^{-1}$ and $I_{j,m(j)}^\omega$ is the basis isometry from $Y_{j,m(j)}^\omega$ onto Y_j . If

$$((X_{M+N,m(M+N)}^\beta, S_{M+N,m(M+N)}^\beta), \dots, (X_{M+N+i,m(M+N+i)}^\beta, S_{M+N+i,m(M+N+i)}^\beta))$$

is a node from the $(\sum_{n=M+N}^{\infty} Y_n)_{c_0}$ -tree where $S_{j,m(j)}^{\beta}$ is defined analogously, then for all $x_j \in X_{j,m(j)}^{\omega}$, $M \leq j \leq M+N-1$, $x_j \in X_{j,m(j)}^{\beta}$, $M+N \leq j \leq M+N+i$, we have for $G \subseteq \{M, \dots, M+N+i\}$

$$\max_{j \in G, \gamma(j)=\omega, \beta} \|x_j\| \|P_{j,m(j)}^{\gamma(j)}\|^{-1} \leq \left\| \sum_{j \in G} x_j \right\| \leq 2CKD \max_{j \in G} \|x_j\|.$$

Thus for each N we have a tree with order $\beta + N$. We can improve the constants by renorming the closed linear span $V^{\beta+\omega}$ of the spaces $X_{n,m}^{\gamma}$, $\gamma \in \{\omega, \beta\}$, $n, m \in \mathbb{N}$, by

$$\|x\|_{\beta+\omega} = \sup \left\{ \|R_{m,n}^{\gamma} (T_{m,n}^{\gamma})^{-1} P_{m,n}^{\gamma} x\| : \gamma = \omega, \beta; m, n \in \mathbb{N} \right\} \vee \frac{\|x\|}{2CKD}.$$

The image of the index $\beta + \omega$ tree in $V^{\beta+\omega}$ has constants 1. By the elastic property $V^{\beta+\omega}$ is K -isomorphic to a subspace of X . Thus X contains a $(\sum_{n=M}^{\infty} Y_n)_{c_0}$ -tree of order $\beta + \omega$ for all $M \in \mathbb{N}$.

For the remaining case we have that α is a increasing limit of a sequence of limit ordinals (β_i) . This step is similar to the case of passing from V^i , $i \in \mathbb{N}$ to V^{ω} that was done at the initial step of the induction. By the induction hypothesis we have spaces V^{β_i} for all $i \in \mathbb{N}$ and for each i a countable family of subspaces $Y_{n,m}^{\beta_i}$, $n, m \in \mathbb{N}$ such that $Y_{n,m}^{\beta_i}$ is isometric to Y_n and there is a $(\sum_{n=N}^{\infty} Y_n)_{c_0}$ -tree in V^{β_i} with all entries in the nodes taken from the subspaces $Y_{n,m}^{\beta_i}$. For each i let T_i be an isomorphism of V^{β_i} into X given by the elastic property. Let $(y_{n,m,k}^{\beta_i})_{k=1}^{\infty}$ be a two-player complementably sequentially reproducible basis of $Y_{n,m}^{\beta_i}$ for each $i, n, m \in \mathbb{N}$. By interweaving the two player games for $(y_{n,m,k}^{\beta_i})_{k=1}^{\infty}$, $i, n, m \in \mathbb{N}$, we can find subsequences $(y_{n,m,k}^{\beta_i})_{k \in K_{n,m}^{\beta_i}}$ of $(y_{n,m,k}^{\beta_i})_{k=1}^{\infty}$ for all $i, n, m \in \mathbb{N}$, such that for all $i, n, m \in \mathbb{N}$, $(T_i y_{n,m,k}^{\beta_i})_{k \in K_{n,m}^{\beta_i}}$ is (equivalent to) a block basis of (w_s) in some order, and for each j, l, p there is a projection $P_{l,p}^{\beta_j}$ from $V^{\alpha} := [(T_i y_{n,m,k}^{\beta_i})_{k \in K_{n,m}^{\beta_i}} : i, n, m \in \mathbb{N}]$ onto $[T_j y_{l,p,k}^{\beta_j} : k \in K_{l,p}^{\beta_j}]$ with $\|P_{l,p}^{\beta_j} y\| \leq \delta \|y\|$ for all $y \in [(T_i y_{n,m,k}^{\beta_i})_{k \in K_{n,m}^{\beta_i}} : i, n, m \in \mathbb{N}, (i, n, m) \neq (j, l, p)]$. Let $R_{n,m}^{\beta_i}$ be the basis equivalence from $(y_{n,m,k}^{\beta_i})_{k \in K_{n,m}^{\beta_i}}$ to $(y_{n,m,k}^{\beta_i})_{k=1}^{\infty}$. By Lemma 13, for each j, l, p we may assume that the projection $P_{l,p}^{\beta_j}$ is zero on $[(T_i y_{n,m,k}^{\beta_i})_{k \in K_{n,m}^{\beta_i}} : i, n, m \in \mathbb{N}, (i, n, m) \neq (j, l, p)]$.

Define a norm on V^{α} by

$$\|y\|_{\alpha} = \sup \left\{ \|R_{n,m}^{\beta_i} T_i^{-1} P_{n,m}^{\beta_i} y\| : i, n, m \in \mathbb{N} \right\} \vee \frac{\|y\|}{KC},$$

and let $B = \sup_{i,n,m} \|R_{n,m}^{\beta_i} T_i^{-1} P_{n,m}^{\beta_i}\|$. As before $\frac{\|y\|}{KC} \leq \|y\|_\alpha \leq B\|y\|$ for all $y \in V^\alpha$. The required properties are easily verified.

This completes the induction step. By Theorem 14 X contains a subspace isomorphic to $(\sum Y_n)_{c_0}$. \square

Now we deduce Theorem 1 from Proposition 15.

Proof. By Theorem 3 of Johnson and Odell we know that an elastic space X with elastic constant K contains a subspace Y K -isomorphic to c_0 . c_0 is isomorphic to $C_0(\omega^n)$ for each $n \in \mathbb{N}$, and thus X has subspaces Y_n K -isomorphic to $C_0(\omega^n)$ for each $n \in \mathbb{N}$. By Proposition 8 and Proposition 15, X has a subspace isomorphic to $(\sum_{n=1}^\infty C_0(\omega^n))_{c_0}$ which is known [2] to be isomorphic to $C_0(\omega^\omega)$.

Inductively, we see that if X contains a subspace K -isomorphic to $C_0(\omega^{\omega^\alpha})$, then because $C_0(\omega^{\omega^\alpha})$ is isomorphic to $C_0(\omega^{\omega^{\alpha n}})$ for each $n \in \mathbb{N}$ [2], by Proposition 8 and Proposition 15, X has a subspace K -isomorphic to $(\sum_{n=1}^\infty C_0(\omega^{\omega^{\alpha n}}))$ which is in turn isomorphic to $C_0(\omega^{\omega^{\alpha+1}})$. For a limit ordinal $\alpha = \lim \beta_n$, a similar argument applied to subspaces of X K -isomorphic to $C_0(\omega^{\omega^{\beta_n}})$ to obtain a subspace isomorphic to $(\sum_{n=1}^\infty C_0(\omega^{\omega^{\beta_n}}))_{c_0}$. The latter space is isomorphic to $C_0(\omega^{\omega^\alpha})$.

Hence X contains subspaces K -isomorphic to $C_0(\omega^{\omega^\alpha})$ for all countable ordinals α . By Theorem 4, X contains a subspace isomorphic to $C[0, 1]$. \square

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